Math 623 Exam 1 Solutions

1. Suppose that $A \in M_n(\mathbb{F})$ has RREF of I_n. Prove that A may be written as the product of elementary matrices.

   See Ex. 0.18. We put A into RREF using elementary matrices $E_1, \ldots, E_k$; i.e. $I = E_k E_{k-1} \cdots E_2 E_1 A$. We then multiply both sides by the same matrix (repeatedly) to get $E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} = A$. The last observation we need is that the inverse of an elementary matrix is elementary, so in fact $E_1^{-1}, \ldots, E_k^{-1}$ are each elementary matrices.

2. Let $J \in M_n(\mathbb{R})$ be the matrix all of whose entries are 1. Find $\sigma(J)$, and for each eigenvalue find a basis for the corresponding eigenspace.

   See Ex 1.1.5. Set $e = (1, 1, \ldots, 1)$; we have $Je = ne$, so (n, e) is an eigenvalue-eigenvector pair. Set $x_i = e - ne_i$; we have $Jx_i = 0 = 0x_i$, so $(0, x_i)$ is an eigenvalue-eigenvector pair. However $\{x_1, \ldots, x_n\}$ is too big (it is dependent, since the sum is zero). Any subset of size $n-1$ will be a basis for the eigenspace corresponding to eigenvalue 0. Note: there are no other eigenvalues since the ones we have found already have total multiplicity n.

3. Give an example of a matrix $M \in M_3(\mathbb{C})$ that is diagonalizable but not diagonal, and has fewer than 3 distinct eigenvalues.

   See Ex. 1.3.9. The simplest approach is to start with a diagonal matrix Λ, then calculate $S\Lambda S^{-1}$ – this is guaranteed to be diagonalizable. We can try something like $\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which yields $M = S\Lambda S^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

4. Calculate the adjugate and eigenvalues of $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

   See Ex. 0.31. This is a straightforward computation; $adj A = \begin{bmatrix} 1 & 1 & -4 \\ -2 & -2 & 8 \\ 1 & 1 & -4 \end{bmatrix}, \sigma(B) = \{-1, 0, 5\}$.

5. Set $P_3(t)$ to be the set of all polynomials of degree at most 3, in variable t, with real coefficients. Find the rank and nullity of linear transformation $T : P_3(t) \to P_3(t)$ given by $T(f(t)) = t\frac{df(t)}{dt}$.

   See Ex. 0.10. A basis for $P_3(t)$ is $\{1, t, t^2, t^3\}$, and we calculate $T(1) = 0, T(t) = t, T(t^2) = 2t^2, T(t^3) = 3t^3$. Hence the range of T is spanned by $\{t, 2t^2, 3t^3\}$; these are clearly linearly independent, hence the rank of T is 3. By the rank-nullity theorem, the rank plus the nullity is the dimension of $P_3(t)$, namely 4. Hence the nullity of T is 1.

6. A matrix $A \in M_3(\mathbb{C})$ is a *square root* of B if $A^2 = B$. Prove that every diagonalizable $B \in M_3(\mathbb{C})$ has a square root.

   See 1.3.P7. Suppose B is diagonalizable; then there is invertible S where $B = SDS^{-1}$, where $D = diag(a, b, c)$. Now, set $E = diag(\sqrt{a}, \sqrt{b}, \sqrt{c})$ (choose either square root if ambiguous), and $A = SES^{-1}$. We calculate $A^2 = SES^{-1}SES^{-1} = SE^2S^{-1} = SDS^{-1} = B$.
   [Note: $\mathbb{C}$ is necessary, else we might not be able to take square roots.]

7. Let $A \in M_3(\mathbb{C})$ be skew-symmetric. Prove that $P_A(t) = -P_A(-t)$, and that if λ is an eigenvalue of A, so is $-\lambda$.

   See 1.4.P2. Since A is skew-symmetric, we have $A = -A^T$. We calculate $P_A(t) = det(tI - A) = det(tI - (-A^T)) = (-1)^3 det(-tI - A^T) = -det((-tI - A^T)^T) = -det(-tI - A) = -P_A(-t)$. Suppose now that (λ, x) is a (right) eigenvalue-eigenvector pair for A. Then $Ax = \lambda x$. We take transposes to get $x^T A^T = \lambda x^T$, then negate to get $x^T(-A^T) = (-\lambda)x^T$ or $x^T A = (-\lambda)x^T$. Hence $(-\lambda, x)$ is a (left) eigenvalue-eigenvector pair for A, and hence $-\lambda$ is an eigenvalue for A.